Skills in Teaching Mathematics

There is no magic formula for teaching mathematics. Each of us struggles to find ways to reach students at levels appropriate for them. As we look for keys, we learn a variety of things that can be catalogued and used later. We talk here about a broad collection of things that are useful in any classroom. They are things for you to try. As you attempt these suggestions, you will compile a list of methods and activities that will help your students learn secondary mathematics.

To teach mathematics, you need to know mathematics. Educators, mathematicians, and others regularly debate how much mathematics you should know. The answer is clouded by bias, opinions, and maybe even a little research.

Not only should you know mathematics, but you need to know a bit about the learning characteristics of the students you will be facing. Your background, general education, and education foundation courses begin to provide you with a basis for this information. That setting should be a platform that continually grows and changes throughout your teaching career.

Knowing mathematics and students provides you with the basics. You need to put those ingredients together into a format that helps you present what you know to students who might not be interested in what they are attempting to study. This last sentence summarizes a plethora of positions, feelings, and opinions you will generate over the years. Furthermore, that sentence is the basis of this entire set of workshops.

Our society is too willing to accept a statement like, “I can’t and don’t like to do math.” The typical response when that statement is made in polite society is, “Yeah, me too.” Thinking in terms of how fast millions of dollars can be sent to the location of a source of inexpensive labor, it is imperative that we teach our students how to use the muscle power between their ears. They are entering a workforce that demands more and more mathematical and thinking flexibility. Thus, an inability to function mathematically (and that mathematics is a lot more than being able to do arithmetic, which calculators do quite well) is critical.

Facing negative attitudes about learning and doing mathematics is a significant challenge to you as a teacher of mathematics. It is your responsibility not only to counteract it, but also to figure out ways to teach mathematics effectively in the face of them. Many of these attitudes are deep-rooted, often based on opinions that started early in the student’s life. On top of that, lack of knowledge about mathematics could well have been where it all started. Given that, you will encounter convictions held by students that are based on the idea that:

All mathematics problems can be solved by application of facts or rules as demonstrated by the teacher or text.

Mathematics problems should be quickly solvable in a few steps.

Mathematics problems can be solved only by methods shown in texts.

The goal in mathematics is to get the right answer.

There is always one best way to do a mathematics problem.

The only mathematics worth knowing is what is tested.

Mathematics students are supposed to receive information and demonstrate that it has been obtained.

The job of a teacher of mathematics is to transmit information to students and check to see that students have gotten it.

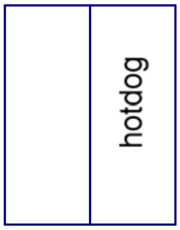
Mathematics is created by very smart people. The rest of us just have to learn what they invent.

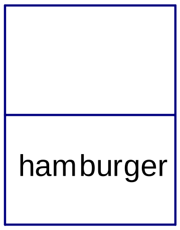
Learning mathematics is based on ability, not how hard you try.

As you prepare lessons, you need to consider these positions, often held by a majority of students in your class.

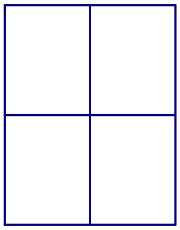
Certainly all of those statements in list in the last paragraph are important. For this discussion, focus on “The goal in mathematics is to get the right answer.” This statement holds the implication that there is always one right answer to a given problem, which is not true, which can be shown through the following activity.

Fold a rectangular sheet of paper in half. (People often ask if they are to fold it hamburger (short and fat), or hotdog (long and skinny) style. It does not matter.)



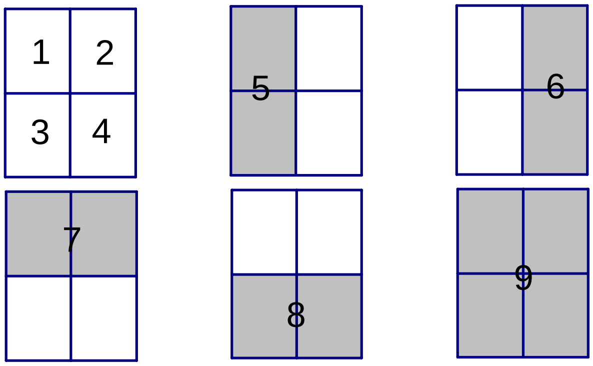


Fold the paper in half again. (People often ask if the second fold should be perpendicular to the first fold, or parallel to it. It does not matter.) Some options for what the folded paper will look like when opened out are:

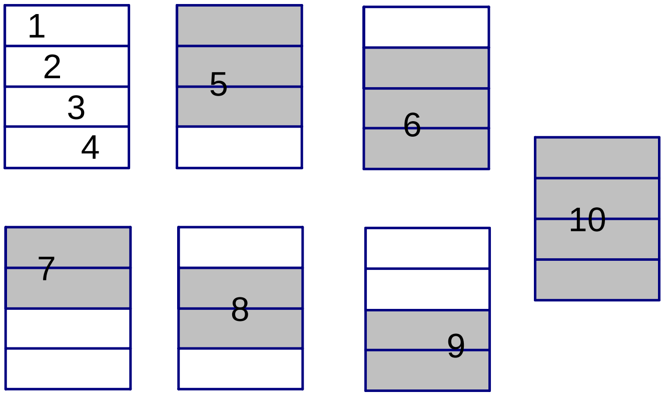


Using only the edges of the paper and the folds you made, how many rectangles are there (if you used line segmented paper, ignore the line segments on it).

The perpendicular folds give a total of 9 rectangles. The shaded sections show the interiors of rectangles that are created by the perpendicular folds and the edges of the paper.



The parallel folds give a total of 10 rectangles. The shaded sections show the interiors of rectangles that are created by the perpendicular folds and the edges of the paper.



If you elect to flip the paper over, the answer changes again. And, if you elect to consider that the paper is about 0.003" thick, there are a whole lot more rectangles formed, there are even more right answers. Certainly the question could have been worded in a manner that would have eliminated any option, but that would have defeated the purpose of the point - - *in **mathematics**, it is possible to have more than one right answer to a problem*.

"WAIT", you say! In the GED program, our mathematics is mostly arithmetic and there, there is always one right answer. OK, consider this.

What is 2 + 2?

More than likely you would say 4.

Well, that depends.

What if you are operating in base 2 (the foundation of computer circuitry)?

In base 2, 4 is written 100. Properly done, 2 would be written as 10 in base 2 and the problem would be written $10_2 + 10_2 = 100_2$. More than likely you are saying that was a dirty trick, but the point is, 4 can be written lots of ways and technically speaking, we often do not call the different ways the same thing. For example, we could write 4 as $\frac{64}{16}$, which would be appropriate if we wanted to know how many sixteenths of an inch are in four inches. So, *even in **arithmetic,** it is possible to have more than one right answer to a problem*. Regretfully, in many arithmetic settings, if the answer to the problem is $\frac{2}{3}$, a response of $\frac{4}{6}$ is deemed unacceptable because it is not "reduced." Actually, what difference does that make? They really are just different names for the same thing, aren't they? Granted, students need to know how to reduce fractions, but value wise, $\frac{4}{6}$ and $\frac{2}{3}$ are the same (and there are an infinite number of other equivalent answers). It is only because of convenience and convention that we say $\frac{2}{3}$ is the best answer. Think in terms of money. If you win $100, would you refuse payment because it is in pennies? Probably not. Sure, it would be more convenient to get a $100 bill, two $50s, five $20s, and so on, but $100 is $100, right?

SOME FACTS ABOUT MATHEMATICS LEARNING.

Most people think the only mathematics needed after high school is arithmetic.

Students get little exposure to problems that require thought before answering.

Students do not acquire skills necessary for learning mathematics on their own.

Students are not acquainted with how mathematics is used in jobs.

Most people are not mathematically prepared for today's jobs, let alone those of tomorrow.

Three out of four Americans stop studying mathematics before completing career prerequisites.

Careers that are growing fastest are those requiring higher mathematics and reasoning capabilities.

Employment opportunities that are not growing rapidly generally require less mathematics and reasoning capability.

HOW DO WE TEACH SKILLS?

Basic skills are necessary for a student to function in the world of mathematics. We all agree that a student needs to be minimally functional at adding, subtracting, multiplying, and dividing whole numbers, fractions, and decimals, for example. There is disagreement with how functional a student must be before a calculator is permitted. At some point the calculator becomes the mode of operation. Prior to that, and after that, there are skills that need to be learned.

In most instances students have been exposed to explanations on how to perform algorithms. For some reason, they are still not functional with the algorithms. It becomes our charge to get them functional. How do we accomplish that task? The students think they know it and are not anxious to pursue the topic again, probably because the drill exercises are often presented in sets that are alike. Drilling on a topic more than likely will involve doing problem sets similar to those used earlier. Little thought is required. A monotonous set of repetitious problems doing the same thing as before holds little appeal.

Why do students object to such treatment? Look at them in an athletic situation. The coach analyzes the game and foundational moves are practiced over and over again. The tennis backhand is hit repeatedly, in an attempt to groove the skill and get a consistent and controllable result. Skateboard skills like jumping a curb and doing a 360 are attempted time after time, striving for that perfect move. In these and so many settings like them, the participant willingly, and often voluntarily, joins in the drill. What are we doing to them in the mathematics class so that similar participation is resisted? Don’t they see the need for these drills? Don’t they realize they will need these skills to be better at the more advanced mathematical levels?

Like the coach, you need to analyze the smallest part of the overall picture. Get the skill clearly sequenced in your mind. Examine the skill from all angles. Consider every possible way a student could do it incorrectly. Then, cover those danger spots as a part of your presentation. Help students overcome the hurdles before they encounter them. For example, consider 800 – 372. The up-front assumption is that students know their subtraction facts. Without that information asking them to do a subtraction problem like this is lunacy. Correctly done, showing the scratch work, the problem would appear as something like:

$$\begin{array}{rccc} & & 9 & \\ & 7 & \cancel{10} & 10 \\ & \cancel{8} & \cancel{0} & \cancel{0} \\ - & 3 & 7 & 2 \\ \hline & 4 & 2 & 8 \end{array}$$

That seems simple enough for us, but we know how to do the problem. For someone who is struggling, there are a lot of rules to remember. Typically we borrow from the number to the left (tens place) when we need more ones so we can subtract. But there is a zero there, so we have to borrow from the hundreds. Forget that one hundred equals 10 tens and an error could happen. Assuming that 10 tens is properly placed, then it is regrouped to be 9 tens and 10 ones. If those 10 ones get put in the right spot, the subtraction can be done and all is well.

Where could a student go wrong? The idea of place value could be ignored and the 10 from the 10 tens placed over the ones zero, leaving the problem to look like:

$$\begin{array}{rccc} & 7 & & 10 \\ & \cancel{8} & 0 & \cancel{0} \\ - & 3 & 7 & 2 \\ \hline & & & 8 \end{array}$$, which becomes $$\begin{array}{rccc} & 6 & & \\ & \cancel{7} & 10 & 10 \\ & \cancel{8} & \cancel{0} & \cancel{0} \\ - & 3 & 7 & 2 \\ \hline & 3 & 3 & 8 \end{array}$$. You might say that seems silly, but remember, you understand the rule. To someone who does not understand the rule, this makes as much sense as anything. This shows the impact of someone trying to memorize a rule with no understanding of what is going on. Mixing steps up makes as much sense as doing them in the right order. The study of error patterns shows how students can do a problem wrong. For more information on error patterns, see Ashlock, R. (2005). *Error Patterns in Computation.* Upper Saddle River, NJ. Prentice Hall. ISBN: 0131198866. While there are many other publications on mathematical error patterns, this one will give you a great start.

Before leaving 800 – 372, take a look at an unusual way to do it that eliminates almost all error potential. The 800 can be written lots of different ways, one of which is 799 + 1. So, rewrite the problem to be:

$$\begin{array}{rcccl} & 7 & 9 & 9 & +1 \\ - & 3 & 7 & 2 & \\ \cline{1-4} \end{array}$$ and all need for regrouping is gone. So, the subtraction is done, giving 427, but do not forget that extra 1 that is to be added, giving an answer of 428. Slick, you say. Why didn't someone show me that? Probably because the teachers you had were singing the same song that had been sung to them, and were not teaching with understanding of the whole picture. Who knows why? But, now you know and you have the opportunity to pass the method on.

Suppose the problem is 804 – 372. The process still works. Rewrite 804 as 799 + 5 and at the end, add 5 rather than 1. If the problem is 841 – 372, it can be rewritten as (799 + 42) – 372, with 42 added at the end. At some point the process becomes more trouble than it is worth and people ask for a simpler way. Guess what! Now they are asking you to explain the rule you have used all of your life, which just so happens to be the rule you were trying to teach them long ago.

There is a multitude of ways of doing arithmetic problems. The rule you use might make sense to you, and that is fine. But, it might not make sense to students, for whatever reason. Ideally, you would know a variety of ways to teach any operation. Consider the partial sum approach to addition, which assumes the student realizes that 500 + 700 is very much like 5 + 7 only the five and seven are not in the ones column. They would also need to be aware of the basics of zero being the identity element for addition on the set of whole numbers. With that background, a problem like 896 + 784 would be done:

$$\begin{array}{rl} 8\,9\,6 & \\ +\,7\,8\,4 & \\ \cline{1-1} 1\,0 & \text{(the sum of 6 and 4)} \\ 1\,7\,0 & \text{(90 + 80, not 9 + 8 as is commonly said)} \end{array}$$

+ 1 5 0 0 (800 + 700, not 8 + 7 as is commonly said)
1 6 8 0

You should notice that the 1 in the sum of 4 and 6 is the regrouped value, as is the case with the 1 in the hundreds column, coming from 8 tens plus 9 tens. As you would do the problem normally, those 1s would be above the tens and hundreds columns, respectively, as your carry.

The *scratch method* for adding whole numbers is another approach you could use.

8 9 6
+ 7 8 4
1 ~~5~~ ~~7~~ 0
6 8

Notice that the 5 is *scratched* out and a 6 is placed below it. Starting from the left, the first sum is 800 + 700, which is 1500. The 15 is written as a part of the sum in the appropriate place values and the zeros are not shown. Then, the addition in the tens column is completed, giving a sum of 17 (really 170). The 7 is written in the tens place of the sum. The 1 from the 170 is really 100 and has to be accounted for. The 100 is shown by scratching out the 5 (which is really 500) and inserting a 6, for 600. If developmental steps were shown, the problem would be:

8 9 6
+ 7 8 4
1 5 0 0 (800 + 700)
1 7 0 (90 + 80)
+ 1 0
1 6 8 0

Here the regrouping "1s" are at the bottom of the hundreds and tens columns, which could prompt the scratch method.

There is another method, developed by Barton Hutchings that can be intriguing and useful for students. Suppose the task is to find the sum of 9, 8, 9, 7, 9 in column addition. Before showing Dr. Huchings' Low Stress Addition consider what is going on in your head as you find this sum. The first part, 9 + 8 is easy because it is a fact Assumed to have been memorized), yielding a sum of 17. The next step is to find the sum of 17 and 9, which is not a fact. Students will have worked this problem but it probably would have been formatted as:

1 (regroup 10 ones as one 10)
1 7
+ 9
2 6. The difficulty is that many students are unable to do a problem such as this mentally. As 17 is added to 9, what really happens in your head is the 17 is expressed as 10 + 7. The "10" is *remembered* and the sum of 7 and 9, which is a fact, is determined to be 16. That 16 is actually 10 + 6 so now the aggregate is 10 + (10 + 6) = (10 + 10) + 6 = 20 + 6 = 26. Granted, you do the

problem much more reflexively than that but if you think about it, that is what is going on in your head.

The Low Stress algorithm eliminates the need to remember all those multiples of ten and keeps the problem as a collection of addition facts. The demonstration below shows the addition of 9 + 8 + 9 + 7 + 9.

		9		
		8		
	1		7	
		9		
	1		6	(7 + 9 = 16)
		7		
	1		3	(6 + 7 = 13)
+		**9**		
	1			(10 from 3 + 9)
		2		(from 3 + 9)
+	4	0		(sum the 10s at the left)
	4	2		(partial sum not normally used here).

This method can be used with more than digit addends. You only need a little more space between columns and any regroup from a value is placed at the top of the respective next column. Students sometimes are confused by a sum like 3 + 4 + 9 when using low stress addition but it reacts like the others.

		3		
		4		
			7	(3 + 4 = 7)
+		9		
1				(10 from 7 + 9)
		6		(6 from 7 + 9)
+	1	0		
	1	6		(partial sum not normally used here)

These are not the only alternate methods for adding whole numbers. The basis for you to provide students with different ways of finding sums is here, though. The rest is up to you. For additional information on alternate algorithms, see Brumbaugh, D. K., Ortiz, E., Gresham, G. (2006). *Teaching Middle School Mathematics*. Mahwah, NJ: Lawrence Erlbaum Associates ISBN: 0-8058-5404-5. OR, go to http://pegasus.cc.ucf.edu/~mathed/chap6.txt and scroll about three fourths of the way down the page. Also, go to http://pegasus.cc.ucf.edu/~mathed/chap7.txt and again go about three fourths of the way down the page.

Connect each skill with some final product. A world-class marathoner does not just get off the couch and run 26.2 miles. There are countless practice hours, many of which are spent developing basics that contribute to the overall picture. One of the best ways to have students learn the required basic skills is to couch the necessary drill in a format that does not appear to focus on the task at hand. Suppose the desire is to provide drill in all four basic operations involving whole numbers. A problem could be presented where the students are asked to use four "threes" and addition, subtraction, multiplication, and division to express counting numbers. For example,

$$1 = \frac{33}{33} \text{ or } \frac{3+3}{3+3} \text{ or } \frac{3+3-3}{3} \text{ or } \frac{3^3}{3^3} \text{ (and others)}$$

$$2 = \frac{3}{3} + \frac{3}{3}$$

$$3 = (3)\left(3^{3-3}\right)$$

$$4 = \left(3^{3-3}\right) + 3$$

$$5 = 3 + \frac{3+3}{3}$$

$$6 = 33 - 3^3$$

Etc.

In the above process, the student does a lot of practicing with the basic operations and will not realize it. The possibility does exist that the activity will not appeal to students. Other number tricks exist that will provide an enticing format. It becomes your task to locate and use them with your students. See Brumbaugh, D., Rock, D. (2001). *Scratch Your Brain C1*. Pacific Grove, CA: Critical Thinking Books and Software. (http://www.criticalthinking.com/index.jsp)

Games are another option. Some games like "Concentration" provide some potential for drill. Another group would be like baseball in which there are different degrees of difficulty questions for singles, doubles, triples, and home runs. Students generally enjoy games like this and they do provide an opportunity to work on skills that need practice.

Many software packages and Web sites involve drill. Abandoning the questions about the sanity of using an expensive computer to serve as a glorified copying machine, this option certainly holds potential. Students are attracted to technology. They quickly become aware of the limitations of drilling on the computer. There is a risk in listing sites for software or activities since they come and go so quickly. Also, they might be considered to be endorsed by the writer. Some are listed here for you to check out. They all have strengths and weaknesses. YOU decide! One huge factor is that many of the software pieces and games seem so childish for adult learners and yet, the activities presented are often mathematically appropriate. Finally, these are not listed in order of preference, quality, strength, weakness, and so on.

www.coolmath4kids.com
www.funbrain.com/
www.aplusmath.com
www.aaamath.com/
http://www.edhelper.com/math.htm

http://www.mathplayground.com/flashcards.html
http://www.allmath.com/flashcards.php
http://math.about.com/od/flashcards/
http://www.mathfax.com/home.htm
www.multiplication.com
http://www.songsforteaching.com/carlsherrill/9bemyfriend.htm
http://mathforum.org/library/topics/basic_ops/

There are records and tape recordings designed to provide basic drill. Consider students who do not know their multiplication facts. Rather than drilling them with flash cards, work sheets, and the like, present them with an opportunity to sing along with a catchy tune that repeats the facts in the music format. It provides an outlet, appeals to the students, and begins the necessary foundations that have been missing. One thing that cannot be forgotten in this—the student is asked for and produces the desired number fact. There should not be a stipulation that the fact must be recalled in a certain manner only.

CONSIDER THE STUDENTS

Look at the people around you the next time you are in a crowd. We come in all sizes, shapes, and colors. These physical differences are easy to spot. Differences in mental capabilities, interests, aptitudes, and attitudes are not easy to ascertain. A study of mathematical achievement showed that a ninth-grade class exhibited a range from third-grade through college-level abilities (Educational Testing Service, 1970). There were students clustered across that continuum. How can a ninth-grade class meet the needs of such a wide variety of levels? The GED classroom has to hold a similar span.

Several Algebra I textbooks were analyzed in 1990 for how they dealt with gender in assignments (Kysilka, 1990). The study concluded that most of the Algebra I texts studied are gender neutral. Some side issues surfaced in the study, however. The research showed that of the six texts examined, one contained 681 word problems and another contained 1,246. Many of the problems were found to be repetitious and boring. A natural question has to focus on how many times we can use money, speed, or time in a word problem before a student begins to desire other settings. Is there a value to investigating a situation involving the time required for eight people to paint a wall rather than seven? Finally, is there a connection between real-world problems and those typically found in texts?

Females and people of different ethnic backgrounds still need to be encouraged to study mathematics. Great strides have been made recently. Not too long ago, the dominant feeling was that “nice girls don’t do mathematics.” Some individuals openly discouraged girls from going into mathematics-related areas. Thankfully, the setting has changed. There is still room for improvement, and we all must struggle to continue the march away from stereotypical classifications and bias. We now have role models.

Sophie Germain (1776–1831) wanted to study and learn but her parents discouraged her. They removed light, heat, and even her clothes to dissuade her. When they found frozen ink she was trying to work with, they figured she was serious about studying. She focused on number theory and analysis and is often referred to as the Hypatia of the 19th century.

Ada Byron Lovelace (1815–1852) suffered from severe headaches and, at the age of 13, her legs became paralyzed. Over the next several years she learned to walk normally. She is credited as being the person to detail the process we now call computer programming.

Grace Brewster Murray Hopper (1906– 1992) designed software over a three-decade span. She was only the third person to work on the Mark I, an 8 feet by 8 feet by 51 feet digital computer. She was in the group that developed the world's first compiler in 1952 and was a pioneer in COBOL. In 1983 she was the oldest officer on active duty in the Navy, becoming the first female rear admiral in 1985.

Evelyn Boyd Granville (1924–) attended Smith College and Yale. She is one of the first female African Americans to earn her doctorate in mathematics. She worked for industry and the government for over 15 years, and was a full professor at California State College in Los Angeles before retiring to Texas to teach mathematics and computer science.

Certainly the list of contributions made by females could be extended. Here is some trivia that might prove useful to you. What do bulletproof vests, fire escapes, windshield wipers, and laser printers have in common? Women invented all of these products.

Encourage each student to see the value of mathematics. This will require energy, and perhaps research, on your part. Determine an interest area of the student and then find mathematical applications there, showing and discussing them with the student. Act excited about learning. If you do not appear excited about the material you are working through with a student, how can you expect them to? Show students applications of the topic being covered. In the process, proceed with caution in some areas: redecoration of a bedroom, taking a trip where you travel so many miles a day, spend so much a day, and so on may not captivate the interest of the student because of lack of practicality in their minds. Talk with a class about a mathless day—no mathematics can be done, and none exists in the world. That means clocks don't exist, cars cannot function, schedules cannot be met, food won't be prepared, and there is no music or video games.

What do we do for the less capable student? Less capable students are not unable to learn. Many who appear less capable are just not excited, and it becomes your responsibility to entice them into the world of mathematics. How is that done? You are a key. You need to believe in the advantages of knowing mathematics and show that to your students. Be excited in the classroom. Show them practical aspects of the things they are learning. Show them how information can be interpreted differently. For example, five incomes in a pool are \$100,000, \$10,000, \$10,000, \$10,000, and \$10,000. The average income is \$28,000, while the median income is \$10,000. These two values are both true, but present a radically different picture of the situation. Students need to be aware of discrepancies like these. Have students show places or careers where they see mathematics used (this might take some prompting from you to give them some examples).

Less capable students often just need an emotional push. Careful sales work by you as you begin the development can overcome most of the obstacles. Suppose the student in question is not able to divide using the standard algorithm. Excluding the calculator may not be the wisest route, but assume that is the mandate, whatever the excuse (excuse is used advisedly here). Showing the student how to do repeated subtraction division might save the day. The standard algorithm demands that all estimations be exact. Repeated subtraction permits estimations below the value needed. Repeated subtraction is easier to visualize. Consider 27401 ÷ 47 and you have a pile of 27,401 sticks. The question is, “How many 47s are in 27,401? Certainly you could remove one group of 47 from the 27,401 sticks, leaving 27,354 sticks. Another batch of 47 could be removed, leaving 27,307 sticks. This process could continue 581 more times and you would learn that a total of 583 sets of 47 are contained in 27,401 sticks. Even the most patient individual would tire of that process before too long.

Using multiples of 10, short cuts begin to appear. Suppose after the second group of 47 is removed, leaving 27,307 sticks, it dawns on the person to remove 10 bunches of 47 or 470 sticks. After that, 2 more sets of 10 bunches of 49 sticks could be removed. At this stage, 32 bunches of 47 sticks have been removed, leaving 25,897. There is still a long way to go.

The idea of multiplying 47 by 10 extends to multiplying by 100 or 1000. Alas, 1000 is too big and is eliminated. Multiple bunches of 47 could be removed 200, or 300 at a time as well. Now old skills come into play because quick mental multiplication of 300 by 47 with 14,100 removed, leaves 11,797 sticks. You can’t take out 300 more bunches of 47 because 14,100 is greater than 11,797, but you could take out 200 bunches of 47, leaving 2397 sticks. You have already done 47 x 300, so 47 x 30 should come quickly and then you could take out 20 more bundles of 47, leaving 47 sticks. You might want to try 50 x 47, which is fine. Either way, there is one bunch of 47 sticks left. The only question remaining is how many bunches of 47 have been removed all together? The answer is 1+1+10+10+10+300+200+50+1 or 583, which is the answer. The significant part of this discussion is that estimates on or below the actual value are acceptable. As estimation skills increase, the repeated subtraction division method collapses to the standard algorithm.

The 27401 ÷ 47 discussion would be shown as:

$$\begin{array}{r r}
47\overline{)27401} & \\
\underline{-\ 47} & 1\\
27354 & \\
\underline{-\ 47} & 1\\
27307 & \\
\underline{-\ 470} & 10\\
26837 & \\
\underline{-\ 470} & 10\\
26367 & \\
\underline{-\ 470} & 10\\
25897 & \\
\underline{-\ 14100} & 300\\
11797 & \\
\underline{-\ \ 9400} & 200\\
2397 & \\
\underline{-\ \ 2350} & 50\\
47 & \\
\underline{-\ 47} & \underline{1}\\
0 & 583
\end{array}$$

The number of bundles of 47 removed is recorded to the right of each removal. The recording could also be above the $\overline{)\ \ }$, which has the advantage of collapsing into the standard algorithm.

```
         1
         1
        10
        10
        10
       300
       200
        50
         1
   _______
47 )27401
     - 47
    27354
     - 47
    27307
    - 470
    26837
    - 470
    26367
    - 470
    25897
  - 14100
    11797
  -  9400
     2397
  -  2350
       47
     - 47
        0
```

As the student’s estimation skills increase, the repeated subtraction approach gets closer and closer to the standard algorithm.

READING AND WRITING IN MATHEMATICS

Reading is an essential tool in a mathematics classroom. A nonreader is handicapped in mathematics, although familiarity with numbers and general formats can conceal many reading weaknesses. Many mathematics students do not realize they cannot read their text like a novel. Reading a mathematics textbook is something like problem solving: Elements need to be selected; items should be placed in a right relation; proper weighting is given to words; and not all words or statements are equal. A major part of reading is related to determining what is and is not important. Problem readers see all pages, paragraphs, and sentences as equally important. We know that is not true. How do we convey that message to students?

One method used to help students read better is the “key-word” approach. They are told that sum means add. If they see sum in any word problem or environment, they know what to do. On the surface that sounds good and it works for many students. But, not all students read well, and not all students attend to words as closely as they should. Consider “Some of the following are odd; which ones? 2, 3, 4, 5, 6, 7, 8, 9.” A student who focuses on key words sees “some.” Spelling is not a big factor and “some” means add. The answer is 44.

You know the difference. You are aware of the need to pay attention. But you are not talking about you. You are talking about students. Couple this reader with the attitude, “If all else fails, read the instructions,” and you begin to have an idea of what you face with some students who are not good readers. It is extremely important that students first comprehend the instructions to the problem they are attempting to solve. In the real world, understanding the situation to be solved is a significant part of the problem. The key-word approach should not be totally abandoned, but caution does need to be used as it is employed.

When you read this sentence, you start at the left and move to the right, word by word to the end of the line, then shift back to the left and start on this line. If you are a speed-reader, you might be able to take in a chunk of the line, or even the entire line at once, but still, the process is similar. That does not apply to something like $\frac{2}{3} + \frac{5}{7}$, where you have to read the 2 and then, rather than reading across the line it is on to the 5, which may or may not be perceived to contain the “+”, you read down. You do not read directly down to the 3, but rather have to interpret the vinculum (line segment below the 2 and above the 3) as meaning division. Then, you read the 3, after which you read diagonally up to the “+” and after interpreting that it means to add, you go on up to the 5 and then repeat the process used for $\frac{2}{3}$. As you know, once you have read the problem, you still have a lot of interpreting and thinking to do before being able to perform the addition. Wonderful things like LCDs, converting to equivalent fractions, remembering to add the numerators only (after they have been expressed with the same denominator), converting to an equivalent mixed number, and remembering to reduce the fraction part if necessary. All of that is a lot more complex than just reading left to right.

You know of the three Rs. Typically, they are not thought of in the sense of complementing each other but, rather, from the standpoint of each standing independent of the other. Certainly each influences the other, and skills from one may be employed in the other. Reading is definitely required in mathematics work, as is writing. However, only recently has there been emphasis on using mathematics to teach and enhance writing skills.

For most students mathematics remains confined between the pages of a textbook or the walls of a classroom and ends with the ringing of the bell. Rarely do students consider mathematics a vital force in history, a part of their daily life, or an essential ingredient for their future. Writing assignments can build an awareness of the extended function of mathematics and its importance in the world.

There are parallels between mathematics work and writing. Understanding the problem is much like outlining a paper. Planning a solution for the problem is similar to organizing a writing project. Actually solving the problem is like writing.

Finally, checking your work is parallel to revising a paper. Actual writing assignments can be incorporated into the mathematics classroom through story problems (creating new ones, rewording others), explanations (tell how or why you have arrived at a given answer or conclusion), paraphrasing (summarize an explanation for another person), or free writing (describe the emotions associated with a topic presented). Writing benefits include increased understanding, more flexible thinking, heightened awareness, better problem-solving skills, practice organizing thoughts, awareness of outside research requirements, information that can be used later, and communications on topics that are not clear.

WHY DISCUSS THE HISTORY OF MATHEMATICS?

Humanity has developed a systematic procedure for storing and passing on information from generation to generation - - history. Much of what is passed on relates to mathematics. Beginning recordings are more mathematical than historical. People chronicled how many long before they recorded events. Even civilizations that developed at different rates and locations exhibit a wide variety of mathematical commonalties. Listing or keeping track of things precedes a system of naming numbers in almost every development. Once numbers are named, rules of operation develop.

The origination of numbers is difficult to pinpoint in history. Early documents from China, Egypt, India, and Mesopotamia all show questions dealing with “how many . . . ?” This implies that the idea of cardinality was around long before the ability to write. This leads to the conclusion that the idea of sets must be one of humanity’s earliest fascinations. As time progressed, body parts and words were used to represent specific numbers. It appears as if it was clear that the emphasis was not on the order in which things were presented but, rather, on the total number. Developmentally, a giant stride forward occurred when it dawned on people that the last cardinal number named also gave a name for the total number of elements of a set. Even with this, it was not long before the number of objects exceeded the names for numbers. Body parts were limited. Another way was needed. This need prompted numeration systems. It must be remembered that spoken and written vocabulary limitations existed and, thus, formalization and extension of a numbering system was not an easy task. Look at the Roman numerals and you will see how it could have been cumbersome to write larger values. If nothing else, it should give an added appreciation for our Hindu Arabic place value system.

The Greeks and the Pythagorean Society did a lot to develop numbers. The Pythagorean Society was open only to aristocrats and all teaching was verbal. Written work would permit secrets to leak out more readily. The Pythagoreans spent a lot of their energy on geometry, but they did develop some good number structure. They developed tables and used the abacus (and we think calculators are new ☺) to do computations. Although they would teach people how to do computations, they would not reveal how the tables were developed. They worked with a wide variety of topics seen today, including perfect squares, triangular numbers, perfect numbers, abundant numbers, deficient numbers, letters representing numbers, primes, and amicable numbers. These investigations involved computations, and that created a demand for some flexible, comprehensive, organized way to write numbers. This, in turn, led to place value.

Through the ages, different bases have been used for place value systems. Although base 10 has become the dominant base, the Duodecimal Society still pushes for base 12. Other bases that were used in the historical development include 20, 60, and 10,000. Imagine trying to get students to memorize operation facts in base 20! As the place value system got refined, computation demands increased.

Computational devices have been available for a long time. Society slowly adapts to new devices and, in the process, provides acceptability. The abacus was probably the first computational device. It is great for addition and subtraction, but the major drawback is that the previous step is consistently eliminated. In 1946, a competition was held between a desk calculator and an abacus. Both operators were equally good with their respective devices. The problems included adding, subtracting, multiplying, and dividing three- to six-digit numbers. The abacus operator easily won the competition.

We take so much for granted. Arithmetic as we know it did not take form until close to the end of the 15th century. Fractions gave mathematicians of antiquity fits. Only in the past 500 or 600 years have fractions been relatively easy to deal with in a number system. Decimals did not appear until the 16th century. Think about it! How could they do some of the computations? It gives an even greater appreciation for the work of those individuals who developed so much of our mathematics. For example, Leibniz completed a computing machine in 1694 and it had a moving carriage, wheels going in opposite directions for addition and subtraction, and latches to prevent over rotation. Babbage created a “difference engine” in 1839, and his son completed the work and published results in 1906. Babbage had the idea, but the technology of the time was too limited to meet his dreams. He needed finer machining tools, electronic circuits, and better alloys. Given these, he had today’s computer. Hollerith developed the idea of holes in cards in 1880. Eventually, IBM adopted this process for use with early computers. Burroughs in 1888 designed a machine that would print figures. Electronics entered the picture in 1944, transistors in 1948. By 1961, computers were taken over by transistors, and the rest is history.

Algebra was available in rudimentary stages from about 1700 B.C. through 1700 A.D. Symbolization developed slowly. Modern symbolism began to emerge in the 14th century. Concepts such as negative numbers were unknown or denied by many mathematicians from antiquity. Realizing these limitations should give you new appreciation for how far we have come in recent years. It certainly gives new meaning to information explosion.

Mathematics as we know it started to be distinguishable only in the 19th century. Mathematics became known not as a tool, or as a descriptor of the world, but as a science. This development was a direct result of more and more people asking "why" something worked out the way it did. These developments brought higher levels of abstractions. Undefined terms were established and more rigorous definitions were devised. There was the understanding that, in response to the realization that as long as terms were loosely defined, varied interpretations would be available. The 20th century found mathematics maturing on all fronts. It expanded in some areas and developed others. Attraction was a unifying force. The study of the history of mathematics can be summarized by the following paragraph, the author of which is unknown:

Many have wondered why we do things in mathematics as we do. Often there is the feeling that so much of what we teach just is, has always been, and will continue to be a ready-made conglomeration of rules and procedures that have come down to us from somewhere, to be used only by the select few that somehow seem to possess a talent for understanding how it all fits together. In short, there is no sense of history behind what we teach, no overview as to how it all began and evolved into the system of topics, concepts, and skills now taught in our courses. However, if topics from the history of mathematics are properly used, and are coupled with an up-to-date knowledge of mathematics and its uses, then it becomes an important tool in the hands of teachers who teach "why." Howard Eves said, "We should let the history of mathematics guide the order in which we present topics to students." (D. K. Brumbaugh, personal communication with H. Eves, Oviedo, FL, 1995)

Howard Eves (1911 – 2004), who was a personal friend of Albert Einstein, spent many winters in the Orlando area. He wrote *An Introduction to the History of Mathematics*, which is referred to by many as THE book about the history of mathematics. Professor Eves and Doug Brumbaugh became friends in the late 1980s. Out of a conversation between them, Professor Eves wrote a math history calendar, which can be found at http://pegasus.cc.ucf.edu/~mathed/Eves. There is an entry for each day of the year that relates to something that happened on that date in the history of mathematics. It is worth your time to take a look at it.

WHAT ABOUT TEXTBOOKS

The essentials in any mathematics textbook have to be similar. Why is this? Why are the textbooks almost generic? Basically, all publishers attempt to create a text that will meet the fundamental objectives as established by different states or curriculum statements published by professional groups. For example, textbooks strive to include ideas brought forward by the Standards (NCTM, 1989) and

Standards 2000 (NCTM, 2000). In doing this, the texts must become similar to meet all the listed objectives.

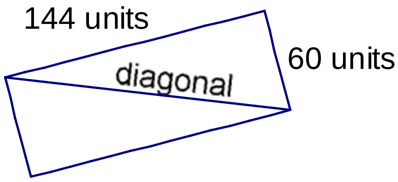
Research indicates that the textbook defines the curriculum taught in most of our schools (Begle, 1973; Fey, 1980; National Advisory Committee on Mathematical Education, 1975; Porter, Floden, Freeman, Schmidt, & Schwille, 1986). Mathematics is a continuum of thoughts and concepts. How much review is sufficient? How much material in a text should be new information? Does the amount of review impact the attitude of students? Are we doing things that turn our students off to the study of mathematics? How much impact does the textbook have on questions like these? Flanders compared three text series from Grades K–9: Addison-Wesley Mathematics (1987) and Algebra (1986), published by Addison-Wesley; Mathematics Today (1985) and HBJ Algebra I (1983), published by Harcourt Brace Jovanovich; and Invitation to Mathematics (1985) and Scott, Foresman Algebra: First Course (1984), published by Scott, Foresman and Company. The D. C. Heath K–8 series follows the trends of the others, but was not reported because there was not an Algebra I book that could be compared with the other three. Each page in each text was reviewed. If there was new material (something that had not been covered before in an earlier grade) anywhere on the page, the page was classified as new, meaning it contained information the student would not have encountered previously. New information could be found in any of the following: lessons, exercises, enrichment activities, or calculator and computer exercises. Even if there was only one challenge exercise on a page that contained something new for the student, that page was classified as new. This gave a liberal interpretation to pages counted as new. Flanders wrote, “In almost all instances, the decision to call a page old or new was easy” (Flanders, 1987, p.19).

Series	Grade	% New
A	6	39
B	6	44
C	6	32
A	7	28
B	7	53
C	7	27
A	8	22
B	8	46
C	8	21
A	9	88
B	9	82
C	9	94

In the above table, the “% new” column represents the part of the text that contained new information. Notice that in books for Grades 6, 7, and 8, only Series B Grade 7 has more than half of the pages containing new information. Furthermore, notice that Series A and C have a steady decline in new information as the grade level increases through Grade 8. Then look at what happens in Grade 9, which was the Algebra I text for the respective series. Talk about culture shock! We lull them to sleep by giving them a lot of the same old material for 3 years and then all of a sudden hit them with a text that has new information on practically every page. This might explain why some of your GED students are not overly interested in learning mathematics.

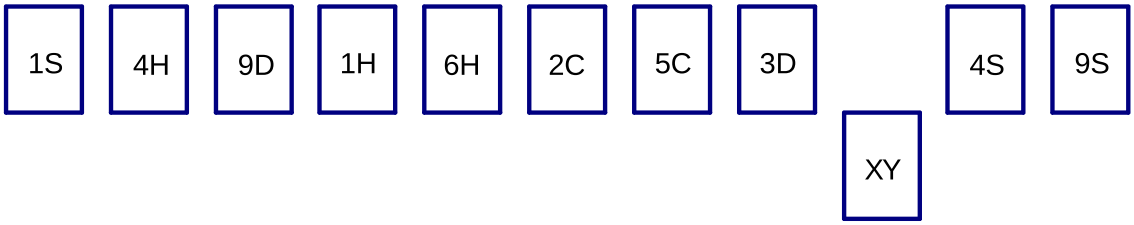
TEACHING FOR GENERALIZATIONS

What is a generalization? What kinds of generalizations are there? How are generalizations used in mathematics? Are there advantages to a student learning generalizations? Part of enticing students to generalize results is getting them to see a problem in a different light. A student is asked to find the length of a rectangle's diagonal given that one side length is 144 units and the other is 60 units.



Certainly, the student could apply the Pythagorean theorem and square both 60 and 144, and then take the square root of the sum of the two squares. More than likely there would be a search for a calculator. By the time it is located and ready to go, the answer could have been derived longhand. However, a different view of the problem makes it even easier to solve. Notice that both 144 and 60 are multiples of 12. Divide out that common factor and the legs of the right triangle are 12 and 5 units respectively. Most students will have seen the 5, 12, 13 right triangle and frequently give the reflexive response of 13. All that remains is to multiply 13 by 12, something that many of them can do in their heads, to get the length of the diagonal of the rectangle to be 156.

Patterning and organization of information often lead to useful generalization skills. Consider this series of cards and determine the numerical value and suit of XY, which has fallen face down.



A hint could be given by putting commas as shown: 1♠, 4♥, 9♦, 1♥ 6♥, 2♣ 5♣, 3♦ XY, 4♠9♠. This should help emphasize the square number pattern 1, 4, 9, 16, 25, 36, 49. Double digits are represented with the same suit, so the missing suit must be diamonds.

This discussion has covered some teaching skills. There are many more. The intent is to start your thinking process toward developing a set of mathematical teaching skills you will reflexively use. The skills you use do not just happen. They are a result of conscious thought, effort, and professionalism. The more energy you devote to developing your teaching skills, the greater will be the benefit for your students.

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